

Almost all quantum states have low entropy rates for any coupling to the environment

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The joint state of a system that is in contact with an environment is called *lazy*, if the entropy rate of the system under *any* coupling to the environment is zero. Necessary and sufficient conditions have recently been established for a state to be lazy [Phys. Rev. Lett. 106, 050403 (2011)], and it was shown that almost all states of the system and the environment do *not* have this property [Phys. Rev. A 81, 052318 (2010)]. At first glance, this may lead us to believe that low entropy rates themselves form an exception, in the sense that most states are far from being lazy and have high entropy rates. Here, we show that in fact the opposite is true if the environment is sufficiently large. Almost all states of the system and the environment are *pretty lazy* – their entropy rates are low for *any* coupling to the environment.

A central question in the study of decoherence and thermalization is how the entropy of a system S changes over time when it is in contact with an environment E [1]. The entropy of the system S is thereby typically measured in terms of the von Neumann entropy $H(S) = -\text{tr}(\rho_S \log \rho_S)$, and quantifies the degree of decoherence of the system [2]. Two extreme cases help to illustrate this measure: If we initially prepare the system in a known pure state, then its entropy is $H(S) = 0$ – no decoherence has yet taken place. However, if the system becomes fully mixed later on all information about its initial state is lost, and at this point its entropy scales with its dimension $H(S) = \log d_S$. To determine the rate of decoherence, i.e. “information loss” over time one is interested in the so-called *entropy rate* [1]

$$\frac{dH(S)}{dt}, \quad (1)$$

of the system evolving according to a coupling Hamiltonian H_{SE}

$$\rho_{SE}(t) = \exp(-iH_{SE}t)\rho_{SE}(0)\exp(iH_{SE}t). \quad (2)$$

Since the von Neumann entropy $H(S)$ also measures the degree of entanglement between the system and the environment, we can equally well think of this quantity as a measure of the rate at which a particular interaction can create entanglement between the system and its environment. Indeed, the value of this derivative at time $t = 0$ is more commonly known in the quantum information community as the *entangling rate* of a particular coupling Hamiltonian H_{SE} [3–5].

How large can this entangling rate be? Intuitively, it is clear that this rate should depend on the interaction strength between the system and the environment. Note that we can write any coupling Hamiltonian as

$$H_{SE} = c\mathbb{I}_{SE} + H_S \otimes \mathbb{I}_E + \mathbb{I}_S \otimes H_E + H_{\text{int}}, \quad (3)$$

where c is a constant. Since the non-interacting terms $H_S \otimes \mathbb{I}_E$ and $\mathbb{I}_S \otimes H_E$ do not contribute to the creation of entanglement between the system and the environment, the interaction strength is often measured in terms of

$\|H_{\text{int}}\|_\infty$. That is, in terms of the largest eigenvalue of H_{int} . A more involved notion of the interaction strength will be introduced later on in the paper. Following [3, 4, 6–9], it has been shown [5] that for any pure state ρ_{SE} and interaction Hamiltonian H_{SE} we have

$$\left| \frac{dH(S)}{dt} \right| \leq c' \|H_{\text{int}}\|_\infty \log d_S, \quad (4)$$

where c' is a constant. For completeness sake, we provide a simple proof for $c' = 4$ in the appendix. This bound is essentially optimal, as it was shown that for any $d_S \leq d_E$ there exists a state with a very large entropy rate. That is, there exists an interaction Hamiltonian H_{SE} such that its entropy rate is $O(\|H_{\text{int}}\|_\infty \log d_S)$, scaling with the dimension of the system d_S .

Are there many states with such high entropy rates? Recent work [1] tackled the problem of studying entropy rates from the other end by providing necessary and sufficient conditions for a state ρ_{SE} to have *zero* entropy rate for *any* Hamiltonian H_{SE} at time $t = 0$ [10]. Such states are also known as *lazy states*. In particular, it was shown that a state ρ_{SE} is lazy if and only if

$$[\rho_{SE}, \rho_S \otimes \mathbb{I}_E] = 0. \quad (5)$$

Lazy states do not have to be eigenstates of H_{SE} or H_{int} , and have several properties that are of interest when it comes to suppressing decoherence. In particular, it was suggested that for a lazy state the entropy of the system could in principle be preserved by fast measurements or dynamical decoupling techniques [11–13].

Yet, lazy states are very unusual. In particular, it was shown [1, 14] using the results of [15] that almost no states are lazy, in the sense that they have measure zero on the joint Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_E$ of the system and the environment [16]. At first glance, this may lead us to believe that low entropy rates themselves are unusual, and that most states should have high entropy rates for at least some coupling Hamiltonian H_{SE} .

RESULT

Here, we show that in fact the opposite is true if the environment is sufficiently large. Almost all states of the system and the environment are “pretty lazy”, that is the entropy rate on the system is very low for any coupling Hamiltonian. With low we thereby mean that the entropy rate scales as some vanishing parameter ε times the interaction strength. Note that in contrast to the study of zero entropy rates, this is all one could hope for when talking about low entropy rates – a stronger interaction strength will necessarily increase any non-zero rate.

Our main result that almost all states have low entropy rates can now be stated slightly more formally. In particular, we will show that the probability that a randomly chosen state ρ_{SE} has large entropy rate is very small. That is,

$$\Pr_{\rho_{SE}} \left[\left| \frac{d}{dt} H(S)_\rho \right| \geq \|H_{\text{int}}\|_\infty \varepsilon \right] \leq \delta, \quad (6)$$

where

$$\varepsilon = 2^{-\frac{1}{2}(\log d_E - 3 \log d_S - 4)}, \quad \delta = 2e^{-d_S^2/16}, \quad (7)$$

and the distribution over the set of possible states on $\mathcal{H}_S \otimes \mathcal{H}_E$ can be any unitarily invariant measure. If the environment is sufficiently large ($\log d_E > 3 \log d_S$) and the system itself is not too small ($\log d_S > 2$), then we obtain a strong statement. We will furthermore show a similar bound that is also interesting for extremely small systems $\log d_S \leq 2$ as long as $\log d_E > (9/2) \log d_S$. In this case, we have

$$\varepsilon = 2^{-\frac{1}{2}(\log d_E - \frac{9}{2} \log d_S - 5)}, \quad \delta = 2e^{-d_S d_E^{1/3}/16}. \quad (8)$$

Since the Hilbert space dimension grows exponentially with the number of constituent particles of a physical system and since we usually assume the environment E to consist of a large number of particles, at least one of the dimensional constraints will be fulfilled in typical situations of physical interest.

It is important to note that while the entropy rate in general depends on the relation between the Hamiltonian and the state (see (12)), the condition for a state being lazy expressed in (5) describes a property of the state alone. Similarly, given the discussed dimensional constraints, the very structure of most states ρ_{SE} is such that they do not allow a fast change of the entropy in S – even for “unphysical” Hamiltonians H_{SE} .

In the appendix, we show that analogous results can be obtained for the linear entropy or purity, which has been studied in the context of entropy rates in [1, 17, 18]. In this case, we even obtain slightly more favourable parameters.

PROOF

Let us now see how we can prove said results. Our proof thereby proceeds in two steps. First of all, we recall that for a randomly chosen pure state from $\mathcal{H}_S \otimes \mathcal{H}_E$ the state will almost certainly be close to fully mixed on \mathcal{H}_S , if the environment is significantly larger than the system [19]. For completeness, we provide a simpler proof of this claim in the appendix. Second, we show that if a state is close to fully mixed on the system \mathcal{H}_S then it is indeed pretty lazy.

Fully mixed on \mathcal{H}_S : Let us first consider only pure states on $\mathcal{H}_S \otimes \mathcal{H}_E$. Note that choosing a random pure state according to the Haar measure is equivalent to applying a randomly chosen unitary U to a fixed starting state, say, $|0\rangle_{SE}$. In contrast to [19] our proof (see appendix) that such a random pure state is fully mixed on the system follows by an easy application of the decoupling theorem [20, 21]. Furthermore, if we apply the decoupling theorem we do not have to restrict to pure states as in [19]. That is, our statement does not only hold for most states of the form $U|0\rangle\langle 0|_{SE}U^\dagger$ but more generally for most states of the form $U\sigma_{SE}U^\dagger$ where σ_{SE} is an arbitrary state (pure or mixed) on $\mathcal{H}_S \otimes \mathcal{H}_E$. Equivalently we may state that most states ρ_{SE} with given eigenvalues and randomly chosen eigenstates are close to fully mixed on the system. “Randomly chosen” here means that the eigenbasis of ρ_{SE} is chosen from the Haar measure, which by definition is unitarily invariant. Since our assertion holds for any fixed set of eigenvalues, it also holds if we pick ρ_{SE} from any unitarily invariant measure on $\mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_E)$, the set of density operators on $\mathcal{H}_S \otimes \mathcal{H}_E$. Summarizing, we obtain the following little lemma, which is proven in the appendix.

Lemma 1. *For a bipartite system $\mathcal{H}_S \otimes \mathcal{H}_E$*

$$\Pr_{\rho_{SE}} \left\{ \left\| \rho_S - \frac{\mathbb{I}_S}{d_S} \right\|_1 \geq \chi \right\} \leq \delta, \quad (9)$$

where the probability is computed over the choice of ρ_{SE} from any unitarily invariant measure on $\mathcal{S}(\mathcal{H}_S \otimes \mathcal{H}_E)$, and where we may choose either

$$\chi = 2^{-\frac{1}{2}(\log d_E - \log d_S - 4)}, \quad \delta = 2e^{-d_S^2/16} \quad (10)$$

or

$$\chi = 2^{-\frac{1}{3}(\log d_E - \frac{3}{2} \log d_S - 5)}, \quad \delta = 2e^{-d_S d_E^{1/3}/16}. \quad (11)$$

Pretty lazy for the von Neumann entropy: Let us now turn to the main part of our proof. A small calculation [1] shows that the rate of change of the von Neumann entropy is given by

$$\frac{dH(S)}{dt} = -i \operatorname{tr} (H_{\text{int}} [\log(\rho_S(t)) \otimes \mathbb{I}_E, \rho_{SE}(t)]) . \quad (12)$$

Note that $[\log(\rho_S) \otimes \mathbb{I}_E, \rho_{SE}] = 0$ if and only if (5) holds, and thus the latter is a sufficient condition for a state ρ_{SE} to be lazy [1]. Consider now a state ρ_{SE} such that its reduced state $\rho_S = \text{tr}_E(\rho_{SE}) = \mathbb{I}_S/d_S$ is fully mixed. Clearly, any such state satisfies (5) and is a lazy state.

How about states which are merely close to being fully mixed on \mathcal{H}_S ? The following lemma captures our intuition that states which are close to lazy states on \mathcal{H}_S are in fact pretty lazy themselves. Closeness is thereby measured in terms of the trace distance [22] which is the relevant quantity for distinguishing quantum states [23].

Lemma 2. *Consider a Hamiltonian with interaction strength $\|H_{\text{int}}\|_\infty$. For any quantum state ρ_{SE} on H_{SE} such that its reduced state is χ -close to fully mixed, i.e., $\chi = \|\rho_S - \mathbb{I}_S/d_S\|_1$ where $\chi \leq 1/d_S$ with $d_S \geq 2$, its entropy rate is bounded by*

$$\left| \frac{dH(S)}{dt} \right| \leq \|H_{\text{int}}\|_\infty 2d_S \chi. \quad (13)$$

Proof. Using (12) we can upper bound the entropy rate by

$$\left| \frac{dH(S)}{dt} \right| \leq \|H_{\text{int}}\|_\infty \|[\log(\rho_S) \otimes \mathbb{I}_E, \rho_{SE}]\|_1 \quad (14)$$

$$\begin{aligned} &= \|H_{\text{int}}\|_\infty \left\| \left[\left(\log(\rho_S) - \log\left(\frac{\mathbb{I}_S}{d_S}\right) \right) \otimes \mathbb{I}_E, \rho_{SE} \right] \right\|_1 \quad (15) \\ &\leq 2 \|H_{\text{int}}\|_\infty \left\| \left(\log(\rho_S) - \log\left(\frac{\mathbb{I}_S}{d_S}\right) \right) \otimes \mathbb{I}_E \right\|_\infty \|\rho_{SE}\|_1 \quad (16) \end{aligned}$$

$$= 2 \|H_{\text{int}}\|_\infty \left\| \log(\rho_S) - \log\left(\frac{\mathbb{I}_S}{d_S}\right) \right\|_\infty, \quad (17)$$

where (14) follows from the fact that for any bounded operators A and B

$$|\text{tr}(AB)| \leq \text{tr}|AB| = \|AB\|_1 \leq \|A\|_1 \|B\|_\infty, \quad (18)$$

(16) follows from the convexity of the L1-norm, and (17) follows from the definition of the L1-norm $\|A\|_1 = \text{tr} \sqrt{A^\dagger A}$. Now let $\{p_i\}_{i=1}^{d_S}$ denote the eigenvalues of ρ_S , so

$$\left| \frac{dH(S)}{dt} \right| \leq 2 \|H_{\text{int}}\|_\infty \cdot \max_{i=1}^{d_S} |\log(p_i d_S)|. \quad (19)$$

We want to maximize the r.h.s. of (19) for fixed

$$\chi = \left\| \rho_S - \frac{\mathbb{I}_S}{d_S} \right\|_1 = \sum_{i=1}^{d_S} \left| p_i - \frac{1}{d_S} \right|. \quad (20)$$

Without loss of generality, let p_1 denote the smallest eigenvalue and p_2 the largest, so $p_1 \leq \frac{1}{d_S} \leq p_2$. The quantity $|\log(p_i d_S)|$ in (19) is monotonously decreasing in p_i if $0 \leq p_i \leq \frac{1}{d_S}$ and monotonously increasing if $\frac{1}{d_S} \leq p_i \leq 1$. The following procedure therefore allows to increase the

r.h.s. of (19) while keeping χ constant: For all $3 \leq i \leq d_S$, if $p_i < \frac{1}{d_S}$ replace $p_1 \mapsto p_1 + p_i - \frac{1}{d_S}$ and $p_i \mapsto \frac{1}{d_S}$. For all $3 \leq i \leq d_S$, if $p_i > \frac{1}{d_S}$ replace $p_2 \mapsto p_2 + p_i - \frac{1}{d_S}$ and $p_i \mapsto \frac{1}{d_S}$. We end up with $p_1 = \frac{1}{d_S} - \frac{\chi}{2}$, $p_2 = \frac{1}{d_S} + \frac{\chi}{2}$, $p_i = \frac{1}{d_S}$ for $3 \leq i \leq d_S$. For $\chi \geq 0$ we have

$$|\log(p_1 d_S)| \geq |\log(p_2 d_S)| \quad (21)$$

so that

$$\left| \frac{dH(S)}{dt} \right| \leq 2 \|H_{\text{int}}\|_\infty \cdot |\log(p_1 d_S)| \quad (22)$$

$$= 2 \|H_{\text{int}}\|_\infty \cdot \left| \log \left(\left(\frac{1}{d_S} - \frac{\chi}{2} \right) d_S \right) \right| \quad (23)$$

$$= 2 \|H_{\text{int}}\|_\infty \cdot \left(-\log \left(1 - \frac{1}{2} d_S \chi \right) \right). \quad (24)$$

Let us now upper bound the term on the r.h.s. Note that for $0 \leq x \leq \frac{1}{2}$ the function $f(x) := -\log(1-x)$ is well defined and convex. By convexity we thus have $f(x) \leq 2f(\frac{1}{2})x = 2x$ on the interval, and hence for $x = (1/2)d_S \chi \leq 1/2$ we have

$$-\log \left(1 - \frac{1}{2} d_S \chi \right) \leq d_S \chi. \quad (25)$$

Upper bounding (24) using (25) now leads to the claimed result. \square

Our claim that almost all states are pretty lazy now follows immediately by combining the two lemmas. Lemma 1 tells us that the probability that a randomly chosen state ρ_{SE} is χ -close to maximally mixed on \mathcal{H}_S is extremely high, where $\chi = 2\sqrt{d_S/d_E}$ and $\chi = 2\sqrt{d_S}/\sqrt[3]{d_E}$ respectively. Lemma 2 now tells us that for sufficiently large d_E such states are indeed pretty lazy. The values for ε in (6) are $d_S \chi$.

Interaction strengths. For completeness, we discuss how our bounds can be improved by a more refined measure of interaction strength. First of all, note that the operators H_S and H_E in (3) are not unique and freedom in their choice may be used to minimize $\|H_{\text{int}}\|_\infty$. Finding the optimal operators H_S and H_E forms in general a non-trivial endeavour. Here, we do not deal with this general task but ask how to best define the “interaction strength” for a given operator H_{int} . Indeed, the decomposition (3) does become unique if we require H_S and H_E to be traceless and H_{int} to have vanishing partial traces on both S and E (e.g. [1, 25]). We can now measure the interaction strength as

$$\Delta(H_{\text{int}}) := 2 \min_{\lambda \in \mathbb{R}} \|H_{\text{int}} - \lambda \mathbb{I}_{SE}\|_\infty. \quad (26)$$

First of all, note that shifting all energy levels of a certain system by a constant amount does not affect the dynamics of that system. These only depend on the *differences*

between the energy eigenvalues. The quantity H_{int} as defined in the decomposition (3) is indeed invariant under addition of a multiple of \mathbb{I}_{SE} to H_{SE} . Similarly, we can see from (12) that adding a multiple of \mathbb{I}_{SE} to H_{int} alone does not affect the rate of change of the local entropy. For this reason, the quantity $\Delta(H_{\text{int}})$ defined in (26) provides a more robust measure of the “interaction strength” of H_{SE} than $\|H_{\text{int}}\|_{\infty}$, as it is already invariant under a shift of eigenvalues in H_{int} . From (26) we furthermore see that this quantity can easily be computed using a semidefinite program (SDP) [27] since we may also write $\Delta(H_{\text{int}}) = 2\gamma$ where γ is the solution of the following SDP

$$\begin{aligned} & \text{minimize} \quad \gamma \\ & \text{subject to} \quad \gamma \mathbb{I} \geq H_{\text{int}} - \lambda \mathbb{I} \geq -\gamma \mathbb{I}, \end{aligned}$$

where the minimization is taken over variables γ and λ . Since $\Delta(H_{\text{int}})$ equals the difference between the smallest and largest eigenvalue of H_{int} we have $\Delta(H_{\text{int}}) \leq 2\|H_{\text{int}}\|_{\infty}$. An upper bound on the entangling rate which is proportional to $\|H_{\text{int}}\|_{\infty}$ may therefore be strengthened by noting that we may replace H_{int} by $H_{\text{int}} - \lambda \mathbb{I}$ without affecting time scales. This allows us to replace $\|H_{\text{int}}\|_{\infty}$ by $\frac{1}{2}\Delta(H_{\text{int}})$ in all the bounds if desired.

DISCUSSION

We have shown that almost all states of the system and the environment are in fact pretty lazy. If the environment E is sufficiently larger than our system S – which we assume to be the case in physical scenarios – the vast majority of bipartite states is such that their entropy in S can only be changed at a vanishing rate. The relevant timescale is thereby given by the inverse of the interaction strength $\|H_{\text{int}}\|_{\infty}$. Our results should be compared to [24, 25] in which it was shown that *equilibration* is a generic property of pure states on $\mathcal{H}_S \otimes \mathcal{H}_E$ if E is sufficiently larger than S . That is, under this conditions almost all joint initial states will lead to the state of S being close to its temporal average for most times. Furthermore it is shown in [25] that for almost all joint initial states, the rate of change of S (the speed of the fluctuations around the temporal average, that is) will on average be small. The time scale that the speed of fluctuations is compared to is here given by $\|H_S \otimes \mathbb{I}_E + H_{\text{int}}\|_{\infty}$. While only H_{int} is able to create entanglement between S and E , both H_{int} and $H_S \otimes \mathbb{I}_E$ are relevant for the evolution of the state of S . If the rate of change of the state of S is low, this implies by Fannes’ inequality [22] that the rate of change of the von Neumann entropy is low as well. So while the results of [24, 25] imply that most initial states will lead to entropy rates on S which in a long-time temporal average are low, we show that most bipartite states really *are* such that the entropy rates on S are low for any interaction Hamiltonian.

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APPENDIX

This appendix is not necessary for the understanding of our work, and merely included for completeness sake.

A word on notation. Let $\mathcal{S}(\mathcal{H}_A)$ denote the set of density operators on system A . For a density operator $\rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$ the min-entropy of A conditioned on B is defined [26] as

$$H_{\min}(A|B)_\rho := \sup_{\sigma_B \in \mathcal{S}(\mathcal{H}_B)} \sup \{ \lambda \in \mathbb{R} : 2^{-\lambda} \mathbb{I}_A \otimes \sigma_B \geq \rho_{AB} \} . \quad (27)$$

For a trivial system B it simplifies to $H_{\min}(A)_\rho = -\log \lambda_{\max}(\rho)$, where λ_{\max} denotes the largest eigenvalue.

Let $|\psi\rangle_{AA'} := \frac{1}{\sqrt{d_A}} \sum_{i=1}^{d_A} |i\rangle_A \otimes |i\rangle_{A'}$ denote the fully entangled state between A and A' . For a CPTPM $\mathcal{T}_{A \rightarrow B}$ (a completely positive and trace-preserving map) we define the Choi-Jamiolkowski representation

$$\tau_{A'B} := (\mathcal{I}_{A'} \otimes \mathcal{T}_{A \rightarrow B})(|\psi\rangle\langle\psi|_{A'A}) \quad (28)$$

where $\mathcal{I}_{A'}$ denotes the identity on $\text{End}(\mathcal{H}_{A'})$. We now first establish an additional lemma that we will use in our proof.

A result from quantum information theory. The following lemma – a corollary of the Decoupling Theorem of [20] – gives a simple characterization of CPTPM's. If the min-entropy $H_{\min}(A'|B)_\tau$ of the Choi-Jamiolkowski representation $\tau_{A'B}$ of a CPTPM as well as the dimension d_A are large, then almost any input state ρ_A will yield an output which is close to τ_B .

Lemma 3. *Let $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ and let $\mathcal{T}_{A \rightarrow B}$ be a CPTPM with Choi-Jamiolkowski representation $\tau_{A'B}$. Then,*

$$\Pr_{U_A} \left\{ \left\| \mathcal{T}_{A \rightarrow B}(U_A \rho_A U_A^\dagger) - \tau_B \right\|_1 \geq 2^{-\frac{1}{2} H_{\min}(A'|B)_\tau} + r \right\} \leq 2e^{-d_A r^2 / 16} \quad (29)$$

where the probability is computed over the choice of U from the Haar measure on the group of unitaries acting on \mathcal{H}_A .

Proof. From [20, Theorem 3.9] with a trivial system R we have for $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ that

$$\begin{aligned} \Pr_{U_A} \left\{ \left\| \mathcal{T}_{A \rightarrow B}(U_A \rho_A U_A^\dagger) - \tau_B \right\|_1 \geq 2^{-\frac{1}{2} H_2(A'|B)_\tau - \frac{1}{2} H_2(A)_\rho} + r \right\} \\ \leq 2e^{-\frac{d_A r^2}{16 K^2 \|\rho_A\|_\infty}} \end{aligned} \quad (30)$$

with $K = \max \{ \|\mathcal{T}(X)\|_1 : X \in \text{Herm}(\mathcal{H}_A), \|X\|_1 \leq 1 \}$. The 2-entropy satisfies $H_2(A'|B)_\tau \geq H_{\min}(A'|B)_\tau$ [20, Lemma 2.3], and similarly $H_2(A)_\rho \geq H_{\min}(A)_\rho \geq 0$. Since $\rho_A \in \mathcal{S}(\mathcal{H}_A)$ we have $\sqrt{\|\rho_A\|_\infty} \leq 1$. Any

$X \in \text{Herm}(\mathcal{H}_A)$ can be written as $X = P_1 - P_2$ with $P_1, P_2 \in \text{Herm}(\mathcal{H}_A)$, $P_1, P_2 \geq 0$. Since \mathcal{T} is trace-preserving and positive (i.e. maps positive operators to positive operators)

$$\begin{aligned} \|\mathcal{T}(X)\|_1 &\leq \|\mathcal{T}(P_1)\|_1 + \|\mathcal{T}(P_2)\|_1 \\ &= \text{tr}[\mathcal{T}(P_1)] + \text{tr}[\mathcal{T}(P_2)] \\ &= \text{tr} P_1 + \text{tr} P_2 \\ &= \|X\|_1 , \end{aligned} \quad (31)$$

so

$$\max \{ \|\mathcal{T}(X)\|_1 : X \in \text{Herm}(\mathcal{H}_A), \|X\|_1 \leq 1 \} \leq 1 . \quad (32)$$

Applying all these inequalities yields the assertion. \square

Proof of Lemma 1.

Proof. Define a CPTPM by $\mathcal{T}_{SE \rightarrow S}(\rho_{SE}) = \rho_S$, i.e. $\mathcal{T}_{SE \rightarrow S} \equiv \text{tr}_E$. Then applying Lemma 3 yields

$$\begin{aligned} \Pr_U \left\{ \left\| \text{tr}_E(U \rho_{SE} U^\dagger) - \tau_S \right\|_1 \geq 2^{-\frac{1}{2} H_{\min}(S'E'|S)_\tau} + \beta \right\} \\ \leq 2e^{-d_S d_E \beta^2 / 16} . \end{aligned} \quad (33)$$

We have $\tau_{S'E'S} = \text{tr}_E |\psi\rangle\langle\psi|_{SES'E'}$ so $\tau_S = \text{tr}_E \frac{\mathbb{I}_{SE}}{d_S} = \frac{\mathbb{I}_S}{d_S}$. The probability is computed over the choice of U from the Haar measure on the group of unitaries on $\mathcal{H}_S \otimes \mathcal{H}_E$. Applying a chain-rule for the min-entropy [26, Lemma 3.1.10] gives

$$H_{\min}(S'E'|S)_\tau \geq H_{\min}(S'E'S)_\psi - \log d_S . \quad (34)$$

It follows directly from the definition of the min-entropy that for a pure state σ_{AB} we have $H_{\min}(A)_\sigma = H_{\min}(B)_\sigma$. Hence

$$\begin{aligned} H_{\min}(S'E'|S)_\tau &\geq H_{\min}(E)_\psi - \log d_S \\ &= H_{\min}(E) \frac{\mathbb{I}_E}{d_E} - \log d_S \\ &= \log d_E - \log d_S . \end{aligned} \quad (35)$$

Inserting this into (33) gives

$$\begin{aligned} \Pr_U \left\{ \left\| \text{tr}_E(U \rho_{SE} U^\dagger) - \frac{\mathbb{I}_S}{d_S} \right\|_1 \geq \sqrt{\frac{d_S}{d_E}} + \beta \right\} \\ \leq 2e^{-d_S d_E \beta^2 / 16} \end{aligned} \quad (36)$$

Choosing $\beta = \sqrt{\frac{d_S}{d_E}}$ we obtain (9) with parameters (10). Alternatively, we choose $\beta = d_E^{-1/3}$ and obtain

$$\begin{aligned} \Pr_U \left\{ \left\| \text{tr}_E(U \rho_{SE} U^\dagger) - \frac{\mathbb{I}_S}{d_S} \right\|_1 \geq \sqrt{\frac{d_S}{d_E}} + d_E^{-1/3} \right\} \\ \leq 2e^{-d_S d_E^{1/3} / 16} \end{aligned} \quad (37)$$

Finally, we use $\sqrt{\frac{d_S}{d_E}} + d_E^{-1/3} \leq 2\sqrt[3]{\frac{d_S}{d_E}}$ to find

$$\Pr_U \left\{ \left\| \text{tr}_E (U \rho_{SE} U^\dagger) - \frac{\mathbb{I}_S}{d_S} \right\|_1 \geq 2\sqrt[3]{\frac{d_S}{d_E}} \right\} \leq 2e^{-d_S d_E^{1/3}/16}. \quad (38)$$

This is (9) with parameters (11). \square

Pretty lazy for the purity. Here, we will extend our results about almost all states being “pretty lazy” to the case where we use the linear entropy or purity as a measure of decoherence instead of the von Neumann entropy. The purity of ρ_S is simply given by $\text{tr}(\rho_S^2)$. The rate of decoherence with respect to this measure is again measured in terms of the time derivative

$$\left| \frac{d}{dt} \text{tr}(\rho_S(t)^2) \right|, \quad (39)$$

where the condition for a zero rate of purity are exactly analogous. I.e., a particular state ρ_{SE} is lazy with respect to purity having zero rate for any interaction Hamiltonian if and only if (5) holds. For this measure of decoherence we have

$$\Pr_{\rho_{SE}} \left[\left| \frac{d}{dt} \text{tr}(\rho_S(t)^2) \right| \geq \|H_{\text{int}}\|_\infty \chi \right] \leq \delta, \quad (40)$$

where we may choose either parameters (10) or (11). With the parameters (10), we only need E to be larger than *one* copy of S in order to obtain a strong statement. Parameters (11) can also be applied in the case of an extremely small system $\log d_S \leq 2$.

Let us now prove those claims. Our argument is essentially analogous to the case of the von Neumann entropy: We already know that most states will be close to maximally mixed on the system, which is itself a lazy state. It thus remains to show that states which are close to such a lazy state are themselves pretty lazy. We obtain a statement very similar to Lemma 2, however this time without an explicit dependence on d_S .

Lemma 4. *Consider a Hamiltonian with interaction strength $\|H_{\text{int}}\|_\infty$. For any quantum state ρ_{SE} on H_{SE} such that its reduced state is χ -close to fully mixed, i.e., $\chi = \|\rho_S - \mathbb{I}_S/d_S\|_1$ where $\chi \leq 1/d_S$ with $d_S \geq 2$, its purity rate is bounded by*

$$\left| \frac{d}{dt} \text{tr}(\rho_S(t)^2) \right| \leq \|H_{\text{int}}\|_\infty \chi. \quad (41)$$

Proof. A brief calculation [1] shows that similarly to (12) the rate of change of the purity is

$$\frac{d}{dt} \text{tr}(\rho_S(t)^2) = i \text{tr} (H_{\text{int}} [\rho_S(t) \otimes \mathbb{I}_E, \rho_{SE}(t)]) . \quad (42)$$

Following the same procedure as in the derivation of (17) we find

$$\left| \frac{d}{dt} \text{tr}(\rho_S(t)^2) \right| \leq \|H_{\text{int}}\|_\infty \|[\rho_S(t) \otimes \mathbb{I}_E, \rho_{SE}]\|_1 \quad (43)$$

$$= \|H_{\text{int}}\|_\infty \left\| \left[\left(\rho_S(t) - \frac{\mathbb{I}_S}{d_S} \right) \otimes \mathbb{I}_E, \rho_{SE} \right] \right\|_1 \quad (44)$$

$$\leq 2 \|H_{\text{int}}\|_\infty \left\| \left(\rho_S(t) - \frac{\mathbb{I}_S}{d_S} \right) \otimes \mathbb{I}_E \right\|_\infty \|\rho_{SE}\|_1 \quad (45)$$

$$= 2 \|H_{\text{int}}\|_\infty \left\| \rho_S(t) - \frac{\mathbb{I}_S}{d_S} \right\|_\infty. \quad (46)$$

Now let $\{p_i\}_{i=1}^{d_S}$ denote the eigenvalues of $\rho_S(t)$. Since $\sum_{i=1}^{d_S} |p_i - 1/d_S| = \chi$ it is clear that

$$\left\| \rho_S(t) - \frac{\mathbb{I}_S}{d_S} \right\|_\infty = \max_{i=1}^{d_S} |p_i - 1/d_S| \quad (47)$$

$$\leq \frac{1}{2} \sum_{i=1}^{d_S} |p_i - 1/d_S| \quad (48)$$

$$= \frac{1}{2} \left\| \rho_S(t) - \frac{\mathbb{I}_S}{d_S} \right\|_1 \quad (49)$$

$$= \frac{1}{2} \chi \quad (50)$$

and hence the assertion. \square

The statement about low purity rates (40) then follows through direct combination of Lemma 1 and Lemma 4.

Upper bound on the entropy rate for arbitrary states and Hamiltonians.

Lemma 5. *For any bipartite Hamiltonian H_{SE} with interaction strength $\|H_{\text{int}}\|_\infty$ we have*

$$\left| \frac{dH(S)}{dt} \right| \leq 4 \|H_{\text{int}}\|_\infty \log d_S. \quad (51)$$

This bound holds for any state ρ_{SE} , pure or mixed, the joint system may be in.

Proof. Let the state of SE be given by ρ_{SE} . Since we did not impose any restrictions on the Hamiltonian whatsoever, we can formally extend the environment with a purifying system P and extend the Hamiltonian to $H_{SEP} = H_{SE} \otimes \mathbb{I}_P$. The interactive part of the Hamiltonian H_{int} gets an additional factor \mathbb{I}_P so that the quantities $\|H_{\text{int}}\|_\infty$ and $\Delta(H_{\text{int}})$ are invariant under this extension.

Let $\rho_{SEP} = |\mu\rangle\langle\mu|_{SEP}$. Then by use of (12) and (18)

$$\left| \frac{dH(S)}{dt} \right| \leq \|H_{\text{int}}\|_\infty \|[\log(\rho_S) \otimes \mathbb{I}_{EP}, |\mu\rangle\langle\mu|_{SEP}]\|_1. \quad (52)$$

Now let $|\nu\rangle_{S\tilde{P}}$ denote a purification of ρ_S . Since both $|\nu\rangle_{S\tilde{P}}$ and $|\mu\rangle_{SE P}$ are purifications of ρ_S , there is an isometry $V_{\tilde{P} \rightarrow EP}$ with $V_{\tilde{P} \rightarrow EP}|\nu\rangle_{S\tilde{P}} = |\mu\rangle_{SE P}$. Hence,

$$\begin{aligned}
& \left| \frac{dH(S)}{dt} \right| \\
& \leq \|H_{\text{int}}\|_{\infty} \left\| \left[\log(\rho_S) \otimes \left(V_{\tilde{P} \rightarrow EP} \mathbb{I}_{\tilde{P}} V_{\tilde{P} \rightarrow EP}^{\dagger} \right), \right. \right. \\
& \quad \left. \left. V_{\tilde{P} \rightarrow EP} |\nu\rangle\langle\nu|_{S\tilde{P}} V_{\tilde{P} \rightarrow EP}^{\dagger} \right] \right\|_1 \\
& = \|H_{\text{int}}\|_{\infty} \left\| V_{\tilde{P} \rightarrow EP} [\log(\rho_S) \otimes \mathbb{I}_{\tilde{P}}, |\nu\rangle\langle\nu|_{S\tilde{P}}] V_{\tilde{P} \rightarrow EP}^{\dagger} \right\|_1 \\
& = \|H_{\text{int}}\|_{\infty} \|[\log(\rho_S) \otimes \mathbb{I}_{\tilde{P}}, |\nu\rangle\langle\nu|_{S\tilde{P}}]\|_1 . \tag{53}
\end{aligned}$$

The commutator may therefore be calculated for an arbitrary purification $|\nu\rangle_{S\tilde{P}}$ of $\rho_S(t)$. The operator $i[\log(\rho_S) \otimes \mathbb{I}_{\tilde{P}}, |\nu\rangle\langle\nu|_{S\tilde{P}}]$ is Hermitian and has vanishing trace, so its eigenvalues are real and sum up to zero. The operator $\Pi_{S\tilde{P}}$ which is the projection onto the eigenstates

with positive eigenvalues therefore allows to write

$$\begin{aligned}
& \|i[\log(\rho_S) \otimes \mathbb{I}_{\tilde{P}}, |\nu\rangle\langle\nu|_{S\tilde{P}}]\|_1 \\
& = 2 \text{tr} \{ \Pi_{SP} i[\log(\rho_S) \otimes \mathbb{I}_{\tilde{P}}, |\nu\rangle\langle\nu|_{S\tilde{P}}] \Pi_{SP} \} \\
& = 2i \text{tr} \{ [\Pi, \log(\rho) \otimes \mathbb{I}] |\nu\rangle\langle\nu| \} \\
& = 2i \langle\nu| [\Pi, \log \rho \otimes \mathbb{I}] |\nu\rangle \\
& \leq 4 |\langle\nu| \Pi (\log \rho \otimes \mathbb{I}) |\nu\rangle| \\
& \leq 4 \sqrt{\langle\nu| \Pi \Pi^{\dagger} |\nu\rangle} \sqrt{\langle\nu| (\log \rho \otimes \mathbb{I})^{\dagger} (\log \rho \otimes \mathbb{I}) |\nu\rangle} \\
& \leq 4 \sqrt{\langle\nu| (\log \rho \otimes \mathbb{I})^2 |\nu\rangle} \\
& = 4 \sqrt{\sum_{i=1}^{d_S} p_i (\log p_i)^2} \\
& \leq 4 \log d_S . \tag{54}
\end{aligned}$$

The second inequality is due to an application of Cauchy-Schwarz, the last one can be proved by use of a Lagrange multiplier. \square